**Class 5: Game Theory**

**General Rules of the Game**

All games discussed here follow these basic rules:

* The game must not be biased (both players have equal power).
* The game must be finite (it must end after a limited number of moves).
* There are two players: **Player 1 (P1)** and **Player 2 (P2)**.
* Both players have equal power — no one gets any special privilege.
* Turns alternate between the two players.
* There are no hidden or player-specific rules.
* The game involves **piles of stones**, each containing a positive number of stones.
* On each turn, a player can remove **any positive number of stones from one single pile**.

**1. Nim Game**

**Description**

* There are several piles of stones.
* On a turn, a player can take any positive number of stones from one pile only.
* The player who takes the last stone wins.
* Player 1 moves first.

**Winning Condition**

* Compute the **XOR-sum** of all pile sizes.
  + If **XOR-sum = 0**, then **P1 loses**.
  + Otherwise, **P1 wins**.

**Technique**

* On each move, Player 1 should try to make the **XOR-sum = 0** after their turn.
* If that’s not possible, then Player 2 can always force the XOR-sum to 0 on their turn, leading to P1’s loss.

**2. Misère Nim**

**Description**

* Same as the Nim Game, except the player who takes the **last stone loses**.
* Player 1 moves first.

**Winning Condition**

* Compute the **XOR-sum** of all pile sizes.
  + If **XOR-sum = 0**, then **P1 loses**.
  + Otherwise, **P1 wins**.

**Special Case**

* If all piles contain exactly **one stone**, then:
  + If **XOR-sum = 0**, **P1 wins**.
  + Otherwise, **P1 loses**.

**Technique**

* Just like Nim, Player 1 tries to make the **XOR-sum = 0** after each of their turns.
* If impossible, Player 2 will do so, ensuring P1’s defeat.

**3. Bogus Nim / Poker Nim**

**Description**

* Similar to the Nim Game, but each player also starts with their own stones.
* On a turn, a player can either:
  + Remove any number of stones from a pile, or
  + Add any number of stones from their own collection into a pile.
* “Bogus” (or “ভুয়া”) means the extra ability doesn’t actually change the game outcome.

**Winning Condition**

* Compute the **XOR-sum** of all pile sizes.
  + If **XOR-sum = 0**, then **P1 loses**.
  + Otherwise, **P1 wins**.

**Technique**

* Player 1 tries to make the **XOR-sum = 0** after their move.
* Adding stones doesn’t affect the result — because the opponent can always take them back optimally.

**Reason**

* The ability to add stones doesn’t create any new advantage.
* The opponent can always respond by removing stones to restore the same XOR balance.

**4. Staircase Nim**

**Description**

* There are several stairs, and each stair has some stones.
* In one move, a player can take any positive number of stones from a single stair and move them to the stair immediately below.
* The game ends when all stones reach stair 0.
* The player who takes the last stone wins.
* Player 1 moves first.

**Winning Condition**

* Consider **1-based indexing** for stairs (i.e., the first stair is stair 1).
* Compute the **XOR-sum** of all stones on **odd-indexed stairs**.
  + If **XOR-sum = 0**, then **P1 loses**.
  + Otherwise, **P1 wins**.

**Technique**

* Stones on even-indexed stairs are **irrelevant (bogus moves)** — they don’t change the game’s outcome.
* Therefore, only the **odd-indexed stairs** affect the winning condition.

**5. Nimble Game**

**Description**

* A row of squares is numbered from 0 to n−1.
* Some squares contain one or more coins.
* On each turn, a player chooses one coin and moves it to any square on its left (a smaller-numbered position).
* Coins cannot move to the right or off the board (below position 0).
* The player who makes the last move wins.

**Winning Condition**

* Treat each coin as a separate Nim pile with size equal to its **position index**.
* Compute the **XOR-sum** of all coin positions.
  + If **XOR-sum = 0**, then **P1 loses**.
  + Otherwise, **P1 wins**.

**Technique**

* On each move, Player 1 should move a coin such that the **XOR-sum becomes 0** after their turn.
* If Player 1 cannot make the XOR-sum 0, then Player 2 can always force it to 0 on their turn and eventually win.

**6. Spruge-Grundy Number**

**7. Green Hagenbush**

a. **Colon Principle**  
b. **Fusion Principle**

**7. Nimble Game**

**Description**

* A row of squares is numbered from 0 to n−1.
* Some squares contain one or more coins.
* On each turn, a player chooses one coin and moves it to any square on its left (a smaller-numbered position).
* Coins cannot move to the right or off the board (below position 0).
* The player who makes the last move wins.

**Winning Condition**

* Treat each coin as a separate Nim pile with size equal to its **position index**.
* Compute the **XOR-sum** of all coin positions.
  + If **XOR-sum = 0**, then **P1 loses**.
  + Otherwise, **P1 wins**.

**Technique**

* On each move, Player 1 should move a coin such that the **XOR-sum becomes 0** after their turn.
* If Player 1 cannot make the XOR-sum 0, then Player 2 can always force it to 0 on their turn and eventually win.